

ON THE GEOMETRY OF WARPED FOLIATIONS

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ABSTRACT. We discuss the geometry of warped foliations. After examining the Levi-Civita connection, and the curvature tensor, we describe the formulae for sectional, Ricci and scalar curvatures. In the final part of this note, we present some examples.

1. INTRODUCTION

The notion of the warped foliation was introduced in [10] by the author of this note. It is a generalization of the M. Berger's modification of a Riemannian structure of S^3 along the fibers of the Hopf fibration called Berger spheres [5].

Warped foliations were widely studied in the point of view of the Gromov-Hausdorff convergence [2]. The sufficient and necessary condition of converging in Gromov-Hausdorff sense of a Riemannian submersion and Riemannian foliation with all leaves compact to the space of leaves with a metric defined by Hausdorff distance of leaves were already developed in [7] and [8]. Moreover, in [9], the author of this paper has presented the connection between the Hausdorff leaf space for a given foliation and the warped foliation.

2. PRELIMINARIES

At the beginning, let us recall the definitions needed in this note.

Let (M, g) be a Riemannian manifold, while ∇ the Levi-Civita connection on M . Let P be a smooth distribution on M . Following [6], one can define a smooth tensor fields T of type $(1, 2)$ by the formula

$$\begin{aligned} g(T(U, V), X) &= g\left(\frac{1}{2}\nabla_U V + \frac{1}{2}\nabla_V U, X\right), \\ g(T(U, X), V) &= -g(T(U, V), X), \\ g(T(X, \cdot), Y) &= g(T(X, \cdot), U) = 0, \end{aligned}$$

and A (also of type $(1, 2)$) by

$$\begin{aligned} g(A(U, V), X) &= g\left(\frac{1}{2}\nabla_U V - \frac{1}{2}\nabla_V U, X\right), \\ g(A(U, X), V) &= -g(A(U, V), X), \\ g(A(X, \cdot), Y) &= g(A(X, \cdot), U) = 0, \end{aligned}$$

where $U, V \in P$, and $X, Y \in P^\perp$. T is called *the second fundamental form of P* , while A *the integrability tensor*. It follows from Frobenius Theorem [3] that for P integrable the integrability tensor A vanishes.

Now, consider a foliation¹ \mathcal{F} on the manifold (M, g) . There are two natural distributions on (M, g) defined by \mathcal{F} . One of them, consisting of all vectors tangent to the leaves of \mathcal{F} is called *tangent*, and will be also denoted by \mathcal{F} . The second one, called *orthogonal*

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¹ Theory of foliations can be found in [1], the amazing book written by A. Candel and L. Conlon

consists of all vectors which are g -orthogonal to the leaves of \mathcal{F} . It will be denoted by \mathcal{F}^\perp .

Denote by T the second fundamental form of \mathcal{F} . Since \mathcal{F} is integrable, its integrability tensor vanishes everywhere. Let S denotes the second fundamental form, while A the integrability tensor of \mathcal{F}^\perp .

Let us recall that a foliation \mathcal{F} satisfying

$$(1) \quad \mathcal{L}_U g(X, Y) = 0,$$

where \mathcal{L} denotes the Lie differentiation, $U \in \mathcal{F}$, $X, Y \in \mathcal{F}^\perp$, is called *Riemannian foliation*[4].

Theorem 2.1. *\mathcal{F} is Riemannian if and only if the second fundamental form S of the orthogonal distribution \mathcal{F}^\perp vanishes.*

A proof can be found in [6].

Throughout this paper, we will also use the following denotations:

$$H_f(U) = \nabla_U \nabla f, \quad h_f(X, Y) = \langle \nabla_X \nabla f, Y \rangle.$$

3. WARPED FOLIATIONS

Let (M, \mathcal{F}, g) be a foliated Riemannian manifold. Consider a smooth function $f : M \rightarrow (0, \infty)$ constant along the leaves of \mathcal{F} . Such function is often called a basic function. Modify the Riemannian structure g to g_f in the following way:

Let $g_f(v, w) = f^2 g(v, w)$ for vectors v, w tangent to the foliation \mathcal{F} . Next, if at least one of vectors v, w is perpendicular to \mathcal{F} then set $g_f(v, w) = g(v, w)$. Foliated Riemannian manifold (M, \mathcal{F}, g_f) we call *the warped foliation* and denote by M_f . The function f is called *the warping function*.

Briefly speaking, one can understand warping as a conformal modification of a Riemannian structure of a foliated Riemannian manifold (M, \mathcal{F}, g) along the leaves of \mathcal{F} with a dilatation equal, for given leaf, to the value of a warping function on this leaf (see Fig. 1). The orthogonal vectors remain unchanged.

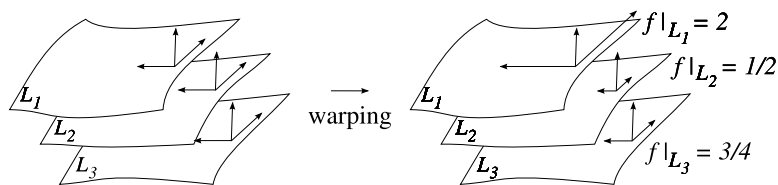


FIGURE 1. Warping of a foliation.

In the following section, we will study the geometry of the warped foliations on a compact Riemannian foliated manifolds. We will calculate the Levi-Civita connection, the curvature tensor, and the curvatures of the warped foliations.

4. LEVI-CIVITA CONNECTION AND CURVATURE TENSOR

Let (M, \mathcal{F}, g) be a foliated Riemannian manifold of dimension p and codimension q . Let ∇ denote the Levi-Civita connection on (M, \mathcal{F}, g) , and let $f : M \rightarrow (0, \infty)$ be a warping function on M . Denote by ∇^f the Levi-Civita connection of the warped foliation M_f .

Theorem 4.1. *The Levi-Civita ∇^f on M_f is given by*

$$\begin{aligned}\nabla_X^f Y &= (\nabla_X Y)^\perp + \frac{1}{f^2}(\nabla_X Y)^\top - \frac{1-f^2}{f^2}A(X, Y), \\ \nabla_U^f V &= (\nabla_U V)^\top + f^2(\nabla_U V)^\perp - \frac{1}{2}\langle U, V \rangle \cdot \nabla f^2, \\ \nabla_X^f U &= \nabla_X U + \frac{1}{2}\frac{Xf^2}{f^2}U - (1-f^2)A(X, U), \\ \nabla_U^f X &= \nabla_U X + \frac{1}{2}\frac{Xf^2}{f^2}U - (1-f^2)A(X, U),\end{aligned}$$

where X and Y are ortoghonal, but U and V tangent to \mathcal{F} .

Proof. By the definition of g_f , and by the Koszul formula

$$\begin{aligned}g_f(\nabla_E F, G) &= \frac{1}{2}(Eg_f(F, G) + Fg_f(G, E) - Gg_f(E, F) \\ &\quad + g_f([E, F], G) - g_f([F, G], E) + g_f([G, E], F)).\end{aligned}$$

Direct calculations give the statement. \square

Following Theorem 4.1, one can calculate the curvature tensor R^f for the warped foliation M_f . Let $W, X, Y, Z \in \mathcal{F}^\perp$ and $P, Q, U, V \in \mathcal{F}$ be vector fields on M . We have

$$\begin{aligned}\langle R^f(X, Y)Z, W \rangle_f &= \langle R(X, Y)Z, W \rangle - 2(1-f^2)\langle A(X, Y), A(Z, W) \rangle \\ &\quad + \langle A(X, W), A(Y, Z) \rangle - \langle A(X, Z), A(Y, W) \rangle \\ &\quad - \frac{1-f^2}{f^2}(\langle S(X, W), S(Y, Z) \rangle - \langle S(X, Z), S(Y, W) \rangle), \\ \langle R^f(X, Y)Z, U \rangle_f &= \langle R(X, Y)Z, U \rangle + f \cdot Xf \cdot \langle A(Y, Z), U \rangle \\ &\quad - f \cdot Yf \cdot \langle A(X, Z), U \rangle - 2f \cdot Zf \cdot \langle A(X, Y), U \rangle \\ &\quad - \frac{Xf}{f}\langle S(Y, Z), U \rangle - \frac{Yf}{f}\langle S(X, Z), U \rangle \\ &\quad - (1-f^2)(\langle (\nabla_X A)(Y, Z), U \rangle + \langle (\nabla_Y A)(X, Z), U \rangle \\ &\quad - \langle A(X, Y), T(U, Z) \rangle),\end{aligned}$$

and

$$\begin{aligned}\langle R^f(X, U)Y, V \rangle_f &= f^2\langle R(X, U)Y, V \rangle + f\langle \nabla_X \nabla f, Y \rangle \langle U, V \rangle \\ &\quad - f \cdot Xf \cdot \langle T(U, V), Y \rangle - f \cdot Yf \cdot \langle T(U, V), X \rangle \\ &\quad + f^2(1-f^2)\langle A(X, V), A(Y, U) \rangle \\ &\quad + (1-f^2)[\langle S(X, V), A(Y, U) \rangle \\ &\quad - \langle S(Y, V), (\nabla_X U)^\perp \rangle - \langle (\nabla_U S)(X, Y), V \rangle]\end{aligned}$$

Moreover, since

$$\langle R^f(U, V)X, Y \rangle + \langle R^f(V, X)U, Y \rangle + \langle R^f(X, U)V, Y \rangle = 0,$$

then

$$\begin{aligned}\langle R^f(U, V)X, Y \rangle_f &= f^2 \langle R(U, V)X, Y \rangle \\ &\quad + f^2(1 - f^2)[\langle A(X, V), A(Y, U) \rangle - \langle A(Y, V), A(X, U) \rangle] \\ &\quad + 2(1 - f^2)[\langle S(X, V), A(Y, U) \rangle - \langle S(Y, V), A(X, U) \rangle].\end{aligned}$$

In addition,

$$\begin{aligned}\langle R^f(U, V)P, X \rangle_f &= f^2 \langle R(U, V)P, X \rangle \\ &\quad + f^2(1 - f^2)\langle A(X, U), T(V, P) \rangle \\ &\quad - f^2(1 - f^2)\langle A(X, V), T(U, P) \rangle \\ &\quad - f(1 - f^2)\langle V, P \rangle \langle A(\nabla f, X), U \rangle \\ &\quad + f(1 - f^2)\langle U, P \rangle \langle A(\nabla f, X), V \rangle \\ &\quad + f\langle U, P \rangle \langle H_f(V), X \rangle - f\langle V, P \rangle \langle H_f(U), X \rangle,\end{aligned}$$

and

$$\begin{aligned}\langle R^f(U, V)P, Q \rangle_f &= f^2 \langle R^\top(U, V)P, Q \rangle \\ &\quad - f^3 \langle U, P \rangle \langle \nabla f, T(V, Q) \rangle - f^3 \langle V, Q \rangle \langle \nabla f, T(U, P) \rangle \\ &\quad + f^4 \langle T(U, P), T(V, Q) \rangle + f^2 \langle U, P \rangle \langle V, Q \rangle \|\nabla f\|^2 \\ &\quad + f^3 \langle V, P \rangle \langle \nabla f, T(U, Q) \rangle + f^3 \langle U, Q \rangle \langle \nabla f, T(V, P) \rangle \\ &\quad - f^4 \langle T(V, P), T(U, Q) \rangle - f^2 \langle V, P \rangle \langle U, Q \rangle \|\nabla f\|^2.\end{aligned}$$

where R denotes the curvature tensor of (M, g) .

5. CURVATURES OF WARPED FOLIATIONS

We are now able to calculate the curvatures for M_f . Let U, V be tangent, while X, Y orthogonal vectors tangent to M_f at point x . The natural consequence of the above section is the following theorem.

Theorem 5.1. *The sectional curvature κ^f of a warped foliations M_f satisfies*

$$\begin{aligned}\kappa^f(X, Y) &= \kappa(X, Y) + 3(1 - f^2)\|A(X, Y)\|^2 \\ &\quad + \frac{1 - f^2}{f^2}\|S(X, Y)\|^2 - \frac{1 - f^2}{f^2}\langle S(X, X), S(Y, Y) \rangle, \\ \kappa^f(X, U) &= \kappa(X, U) - \frac{1}{f}h_f(X, X) + 2fXf\langle T(U, U), X \rangle \\ &\quad - (1 - f^2)[\langle (\nabla_U S)(X, X), U \rangle - \|S(X, U)\|^2] \\ &\quad - f^2(1 - f^2)\|A(X, U)\|^2, \\ \kappa^f(U, V) &= \frac{\hat{\kappa}(U, V)}{f^2} - \frac{\|\nabla f\|^2}{f^2} - f^4\langle T(U, U), T(V, V) \rangle \\ &\quad + f^4\|T(U, V)\|^2 + f\langle \nabla f, T(V, V) \rangle + f\langle \nabla f, T(U, U) \rangle,\end{aligned}$$

where $\hat{\kappa}$ denotes the sectional curvature of a leaf.

Proof. Follows directly from the formulae for curvature tensor from the previous section. \square

We will now calculate the Ricci tensor, and the Ricci curvature for the warped foliation M_f . Let $U_1, \dots, U_p, X_1, \dots, X_q$ be an orthogonal basis on M_f in a point x . Let us recall, that for any $E, F \in T_x M_f$ we have

$$\text{Ric}^f(E, F) = \sum_{i=1}^p \langle R^f(U_i, E)F, U_i \rangle_f + \sum_{j=1}^q \langle R^f(X_j, E)F, X_j \rangle_f,$$

where $\langle \cdot, \cdot \rangle_f = g_f(\cdot, \cdot)$. Set $\bar{U}_i = fU_i, i = 1, \dots, p$. $\bar{U}_1, \dots, \bar{U}_p, X_1, \dots, X_q$ form orthogonal basis in $T_x M$. By the results of Section 4, for any U, V tangent to \mathcal{F}

$$\begin{aligned} \langle R^f(U_i, U)V, U_i \rangle_f &= \langle R^\top(\bar{U}_i, U)V, \bar{U}_i \rangle \\ &\quad - f^2 \langle T(\bar{U}_i, \bar{U}_i), T(U, V) \rangle + \langle T(\bar{U}_i, U), T(\bar{U}_i, V) \rangle \\ &\quad - \langle U, V \rangle \|\nabla f\|^2 + \langle \bar{U}_i, U \rangle \langle \bar{U}_i, V \rangle \|\nabla f\|^2 \\ &\quad + f \langle T(U, V), \nabla f \rangle + f \langle U, V \rangle \langle T(\bar{U}_i, \bar{U}_i), \nabla f \rangle \\ &\quad + f \langle \bar{U}_i, U \rangle \langle T(\bar{U}_i, V), \nabla f \rangle + f \langle \bar{U}_i, V \rangle \langle T(\bar{U}_i, U), \nabla f \rangle \end{aligned}$$

and

$$\begin{aligned} \langle R^f(X_j, U)V, X_j \rangle_f &= f^2 \langle R(X_j, U)V, X_j \rangle \\ &\quad + (1 - f^2) [\langle S(X_j, U), S(X_j, V) \rangle + \langle \nabla_U S(X_j, X_j), V \rangle] \\ &\quad + f^2 (1 - f^2) \langle A(X_j, U), A(X_j, V) \rangle \\ &\quad - 2f X_j f \langle T(U, V), X_j \rangle \\ &\quad + f \langle \nabla_{X_j} \nabla f, X_j \rangle \langle U, V \rangle. \end{aligned}$$

Recall that

$$\text{Ric}^f(U, V) = \sum_{i=1}^p \langle R^f(U_i, U)V, U_i \rangle_f + \sum_{j=1}^q \langle R^f(X_j, U)V, X_j \rangle_f.$$

Finally

$$\begin{aligned} (2) \quad \text{Ric}^f(U, V) &= \text{Ric}^\mathcal{F}(U, V) + f^2 \text{Ric}^\perp(U, V) \\ &\quad - f^2 \langle H^\mathcal{F}, T(U, V) \rangle + f \langle U, V \rangle \langle H^\mathcal{F}, \nabla f \rangle \\ &\quad - (p-1) \langle U, V \rangle \|\nabla f\|^2 + pf \langle T(U, V), \nabla f \rangle \\ &\quad + f^2 \langle T^\top U, T^\top V \rangle - f^2 (1 - f^2) \langle A^\perp U, A^\perp V \rangle \\ &\quad + (1 - f^2) \langle S^\perp U, S^\perp V \rangle + (1 - f^2) \text{tr}^\perp \langle \langle \nabla_U S \rangle(\cdot, \cdot), V \rangle \\ &\quad - f \langle U, V \rangle \text{tr}^\perp h_f \end{aligned}$$

where

$$\begin{aligned}
\langle T^\top U, T^\top V \rangle &= \sum_{i=1}^p \langle T_{U_i} U, T_{U_i} V \rangle, \\
\langle A^\perp U, A^\perp V \rangle &= \sum_{i=1}^q \langle A_{X_i} U, A_{X_i} V \rangle, \\
\langle S^\perp U, S^\perp V \rangle &= \sum_{i=1}^q \langle S_{X_i} U, S_{X_i} V \rangle, \\
H^\mathcal{F} &= \sum_{i=1}^p T(U_i, U_i), \\
\text{tr}^\perp F(\cdot, \cdot) &= \sum_{i=1}^q F(X_i, X_i).
\end{aligned}$$

Now, let X, Y be orthogonal to \mathcal{F} . Similarly, we get

$$\begin{aligned}
(3) \quad \text{Ric}^f(X, Y) &= \text{Ric}^\perp(X, Y) + \text{Ric}^\top(X, Y) \\
&\quad - (1 - f^2) \langle A_X^\top, A_Y^\top \rangle + 3(1 - f^2) \langle A^\perp X, A^\perp Y \rangle \\
&\quad + \frac{Xf}{f} \langle H^\mathcal{F}, Y \rangle + \frac{Yf}{f} \langle H^\mathcal{F}, X \rangle \\
&\quad - p \frac{h_f(X, Y)}{f} + \frac{1 - f^2}{f^2} \langle H^\perp, S(X, Y) \rangle \\
&\quad - \frac{1 - f^2}{f^2} [\langle S_X^\top, A_Y^\top \rangle - \text{tr}^\top \langle (\nabla \cdot S)(X, Y), \cdot \rangle \\
&\quad - \langle S_Y^\top, (\nabla_X^\perp)^\top \rangle - \langle S_X^\perp, S_Y^\perp \rangle],
\end{aligned}$$

with

$$\begin{aligned}
H^\perp &= \sum_{i=1}^q T(X_i, X_i), \\
\langle S_X^\top, A_Y^\top \rangle &= \sum_{i=1}^p \langle S(X, U_i), A(Y, U_i) \rangle, \\
\text{tr}^\top \langle (\nabla \cdot S)(X, Y), \cdot \rangle &= \sum_{i=1}^p \langle (\nabla_{U_i} S)(X, Y), U_i \rangle, \\
\langle S_Y^\top, (\nabla_X^\perp)^\top \rangle &= \sum_{i=1}^q \langle S(Y, U_i), \nabla_X^\perp U_i \rangle, \\
\langle S_X^\perp, S_Y^\perp \rangle &= \sum_{i=1}^q q \langle S(X, X_i), S(Y, X_i) \rangle.
\end{aligned}$$

Finally, if X is orthogonal and U tangent to \mathcal{F} , we have

$$\begin{aligned}
 (4) \quad \text{Ric}^f(X, U) &= \text{Ric}^\perp(X, U) + \text{Ric}^\top(X, U) \\
 &+ 3f\langle A(\nabla f, X), U \rangle + (p-1)\frac{(1-f^2)}{f}\langle A(\nabla f, U), X \rangle \\
 &- \frac{p-1}{f}h_f(U, X) + (1-f^2)\langle A_X^\top, T_U^\top \rangle \\
 &+ (1-f^2)\langle A(U, X), H^\mathcal{F} \rangle + \frac{1}{f}\langle S(X, \nabla f), U \rangle \\
 &+ (1-f^2)\text{tr}^\perp\langle (\nabla \cdot A)(X, \cdot) - (\nabla_X A)(\cdot, \cdot), U \rangle \\
 &- \frac{Xf}{f}\langle H^\perp, U \rangle,
 \end{aligned}$$

where $(\nabla \cdot A)(X, \cdot) = \sum_{i=1}^q (\nabla_{X_i} A)(X, X_i)$.

Let $E \in T_x M_f$ be a unit vector. We have $E = aU + bX$, where $|U|_f = |X|_f = 1$, $U \in \mathcal{F}^\top$, $X \in \mathcal{F}^\perp$ and $a^2 + b^2 = 1$. Note that Ricci curvature ric^f in a point x for the warped foliation M_f is given by the formula

$$\begin{aligned}
 \text{ric}^f(E) &= \text{Ric}^f(E, E) \\
 &= a^2 \text{Ric}^f(U, U) + 2ab \text{Ric}^f(X, U) + b^2 \text{Ric}^f(X, X).
 \end{aligned}$$

Theorem 5.2. *The Ricci curvature ric^f in a point x for the warped foliation M_f satisfies*

$$\begin{aligned}
 \text{ric}^f(E) &= a^2(\text{Ric}^\mathcal{F}(U, U) + f^2 \text{Ric}^\perp(U, U)) + 2ab(\text{Ric}(X, U) + b^2 \text{Ric}(X, X)) \\
 &+ a^2(-f^2\langle H^\mathcal{F}, T(U, U) \rangle + \frac{1}{f}\langle H^\mathcal{F}, \nabla f \rangle - \frac{(p-1)\|\nabla f\|^2}{f^2}) \\
 &+ pf\langle T(U, U), \nabla f \rangle + f^2\|T^\top U\|^2 - f^2(1-f^2)\|A^\perp U\|^2 \\
 &+ (1-f^2)\|S^\perp U\|^2 + (1-f^2)\text{tr}^\perp\langle (\nabla_U S)(\cdot, \cdot), U \rangle - \frac{\text{tr}^\perp h_f}{f}) \\
 &+ 2ab(3f\langle A(\nabla f, X), U \rangle + (p-1)\frac{(1-f^2)}{f}\langle A(\nabla f, U), X \rangle \\
 &- \frac{p-1}{f}h_f(U, X) + (1-f^2)\langle A_X^\top, T_U^\top \rangle + (1-f^2)\langle A(U, X), H^\mathcal{F} \rangle \\
 &+ \frac{1}{f}\langle S(X, \nabla f), U \rangle + (1-f^2)\text{tr}^\perp\langle (\nabla \cdot A)(X, \cdot) - (\nabla_X A)(\cdot, \cdot), U \rangle \\
 &- \frac{Xf}{f}\langle H^\perp, U \rangle) \\
 &+ b^2(-(1-f^2)\|A_X^\top\|^2 + 3(1-f^2)\|A^\perp X\|^2 + \frac{2Xf}{f}\langle H^\mathcal{F}, X \rangle \\
 &- p\frac{h_f(X, X)}{f} + \frac{1-f^2}{f^2}\langle H^\perp, S(X, X) \rangle - \frac{1-f^2}{f^2}[\langle S_X^\top, A_X^\top \rangle \\
 &- \text{tr}^\top\langle (\nabla \cdot S)(X, X), \cdot \rangle - \langle S_X^\top, (\nabla_X^\perp)^\top \rangle - \langle S_X^\perp, S_X^\perp \rangle]).
 \end{aligned}$$

Proof. The proof follows directly from the formulae (2)-(4). \square

Finally, we can formulate how the scalar curvature changes while the foliation is warped by a function f .

Theorem 5.3. *The scalar curvature s^f of the warped foliation M_f satisfies*

$$\begin{aligned}
s^f &= s^{\mathcal{F}} + f^2(s^\top)^\perp + s^\perp \\
&\quad - f^2\|H^{\mathcal{F}}\|^2 + p\frac{1}{f}\langle H^{\mathcal{F}}, \nabla f \rangle - \frac{p(p-1)\|\nabla f\|^2}{f^2} \\
&\quad + pf\langle H^{\mathcal{F}}, \nabla f \rangle + \sum_{j=1}^p (f^2\|T^\top U_j\|^2 - f^2(1-f^2)\|A^\perp U_j\|^2 \\
&\quad + (1-f^2)\|S^\perp U_j\|^2 + (1-f^2)\text{tr}^\perp(\langle \nabla_{U_j} S \rangle(\cdot, \cdot), U_j)) - p\frac{\text{tr}^\perp h_f}{f} \\
&\quad - \sum_{i=1}^q ((1-f^2)\|A_{X_i}^\top\|^2 + 3(1-f^2)\|A^\perp X_i\|^2) + \frac{2}{f}\langle H^{\mathcal{F}}, \nabla f \rangle \\
&\quad - p\frac{\text{tr}^\perp h_f}{f} + \frac{1-f^2}{f^2}\langle H^\perp, H^\perp \rangle - \sum_{i=1}^q \left(\frac{1-f^2}{f^2} [\langle S_{X_i}^\top, A_{X_i}^\top \rangle \right. \\
&\quad \left. - \text{tr}^\top(\langle \nabla \cdot S \rangle(X_i, X_i), \cdot) - \langle S_{X_i}^\top, (\nabla_{X_i}^\perp)^\top \rangle - \langle S_{X_i}^\perp, S_{X_i}^\perp \rangle] \right).
\end{aligned}$$

where $s^\top)^\perp = \sum_{i=1}^q \text{Ric}^\perp(U_i, U_i)$.

Proof. It follows directly from the formula

$$s^f(x) = \sum_{i=1}^m \text{ric}^f(E_i),$$

and Theorem 5.2. □

6. EXAMPLES

We now will study some examples of warped foliations and its curvatures.

Let (M, g) be a compact 2-dimensional foliated manifold carrying a 1-dimensional foliation \mathcal{F} . Let suppose that the sectional curvature of the manifold $\kappa = 0$. Let $(f_n)_{n \in \mathbb{N}}$ be a sequence of constant warping functions such that $f_n = \frac{1}{n}$.

Theorem 6.1. $\lim_{n \rightarrow \infty} \kappa^{f_n} = -\langle (\nabla_S)(X, X), U \rangle + \|S(X, U)\|^2$.

Proof. By Theorem 5.1,

$$\begin{aligned}
\kappa^f(X, U) &= \kappa(X, U) - \frac{1}{f}h_f(X, X) + 2fXf\langle T(U, U), X \rangle \\
&\quad - (1-f^2)[\langle (\nabla_U S)(X, X), U \rangle - \|S(X, U)\|^2] \\
(5) \quad &\quad - f^2(1-f^2)\|A(X, U)\|^2,
\end{aligned}$$

where X and U are vectors orthogonal and tangent to \mathcal{F} , respectively. Since \mathcal{F} is a foliation of codimension one, the integrability tensor A of the orthogonal distribution vanishes everywhere. Moreover, $Ef = 0$ for any vector field E on M . Again,

$$\kappa^{f_n}(X, U) = \kappa(X, U) - (1-f^2)[\langle (\nabla_U S)(X, X), U \rangle - \|S(X, U)\|^2].$$

Recall that $\kappa(X, U) = \kappa = 0$, and $f_n \rightarrow 0$. Finally,

$$\lim_{n \rightarrow \infty} \kappa^{f_n} = -\langle (\nabla_S)(X, X), U \rangle + \|S(X, U)\|^2.$$

This ends our proof. □

Corollary 6.1. *The sectional curvature of a warped by constant functions 1-dimensional Riemannian foliation on a compact 2-dimensional Riemannian manifold of curvature equal to zero is constant, and remains zero.*

Proof. By Theorem 2.1, the second fundamental form S vanishes everywhere. \square

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